Gravitational Theory with Local conservation of Energy

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ABSTRACT

The presentation here is based on the presumption that the total energy of a particle and photon in a gravitational field is localized and conserved. A mass particle thus entering a static gravitational field has an increasing velocity, but a decreasing rest mass, or a mass defect. The total energy is conserved. This also means that as a photon rises in a gravitational field there is no loss of energy, and therefore a photon escapes the most intense field, precluding the formation of a black hole. Since there is no energy change in an accelerating particle **technically gravitation is not a force**. It will be shown that such a theory of gravitation can be developed, that properly predicts known dynamic, has proper covariant transformations, the proper Shapiro velocity, and does not require formulation in curved space. Noether's theorem definitively shows that contrary to all other forces, energy cannot be conserved nor localized in a Riemannian gauge field representation. It is presumed here that this is a flaw in GR, and it is asserted here that Noether's theorem is not an indicator of a physical reality, but an indicator of the approximate nature of GR. This can best be tested in the observation of the properties of objects cited to be black holes. There are points of this development that are testable, and provable or disprovable in experiments on Black Holes and, Event horizons.

INTRODUCTION

As is well known, but presumed unimportant, are aspects of the Ricci tensor representation that illustrate the theory is an approximation to the correct representation, but is not complete or accurate representation.

The obvious shortfalls are that:

1. Mass is represented as a continuous density function, when reality requires discrete point functions, and there is insufficient complexity in the tensor expressions to represent mass as a collection of point particles.

2. GR does not scale down, nor properly function at microscopic level, though the theory recognizes no scaling limits.

3. It is a gauge field, with an infinite number of infinitesimal generators. Because of this, Noether's theorem illustrates that energy tensor is not covariant under general coordinate transformations, and there can be no local conservation of energy, meaning the flow of energy in and out of a spacetime volume is not conserved [9]. This leads directly the concept of black holes, since the source of the kinetic energy gained by a particle entering a gravitational field comes from the field and not the rest mass. There is no rest mass defect.

4. It is not covariant under general coordinate transformations. Local energy balance is dependent on the coordinates used for the calculation; consequently different results are obtained for different coordinate frames [4].

5. The distance between any two points in the defined curved space is ambiguous, and dependent on the path [4].

Researchers who do not view GR as an approximation do not consider these issues shortcomings, but the reality of physics.

The dynamic particle interactions presented here are formulated in covariant differential and algebraic relations. It is shown that the phenomenology of GR can be reproduced without resorting to Riemannian space curvature and does not result in unphysical singularities. This development will adhere to a flat Minkowski space ($\delta\Delta t = 0$) and a variable speed of light.

The total energy of a particle is presumed localized in the volume of the particle, and as the particle accelerates in the field the kinetic energy increases at expense of the rest energy. There is thus no work done on the particle and technically the field is not a force, as work done on the particle would increase energy. The technical issues involved in the acceleration is more fully in developed in" The Concept of Mass as Interfering Photons, and the Originating Mechanism of Gravitation"[1].

Current views of photon energy in GR are contrary to Einstein's original view that the photon energy is constant and the shift is due slower clock associated with the emission. [12]. Current views of GR are that the photon loses energy to the field on rising.

This development is cast in Minkowski space, and ascribes the frequency of a photon rising in a gravitational field to have a lower measured frequency at the elevated receptor due to the lower rest mass of the emitter not a change in the time scale or a loss of energy. After taking account of the rest mass change at different elevations, the results of the Pound-Rebka-Snider experiment shows that the energy of the photon is conserved. [6]

GENERAL DEVELOPMENT

For a massive particle in a gravitational potential, the total mass of a particle at rest relative to an observer external to the field is defined to be:

$$M^2 = M_0^2\left(1-\frac{\mu}{r}\right)^2 \quad , \qquad (1.1)$$

where M_0 is the rest mass external to the gravitational potential. The relativistic mass is then:

$$M^2\left(1-\frac{v^2}{c^2}\right)=M_0^2\left(1-\frac{\mu}{r}\right)^2 \quad (1.2)$$

This relation Eq.(1.2), is presumed to be the fundamental relation between mass particles gravity and velocity or the primary postulate of the theory. With this and local conservation of energy, all the other dynamical relations can be derived including the Shapiro velocity of light.

GR presumes the fundamental postulate to be the Ricci tensor curvature of spacetime which is a far more expansive assumption, and can only deal with aggregates of matter having no microscopic scaling for point particles

Though similar to standard Lagrangian expressions Eq.(1.2), is a 2^{nd} order departure but is easy to show that it can be cast in the form of the well known Lagrangian within measurable accuracy. i.e.

$$Mc^2=\left(M_0c^2-\frac{GMm}{r}+\frac{1}{2}Mv^2\right) \quad (1.3)$$

Orbital Mechanics

In the following it will be shown that the orbital precession predicted by GR can calculated with Eq.(1.2), without resorting to curved space, only including known Special Relativistic considerations.

Most calculations of orbital motion tend to neglect issues related to the makeup of the potential term.

$$\phi=\frac{GMm}{r}, \quad (1.4)$$

First is that the mass terms have to be the relativistic mass. This is obvious from the fact that, if the particles happen to be spinning the kinetic energy must be included, meaning the mass is relativistic mass. In addition each mass experience the other as if it is moving with their relative velocity.

From our knowledge of the Thomas precession, it is known that the distance a particle traveling the circumference of a circle around an attracting potential is shortened by the relativistic contraction. We would assert that if the circumference of a circle is contracted as the result of the relativistic velocity, the radius must also be contracted.

With those considerations the gravitation term in Eq.(1.3) , should be:

$$\frac{GMm}{r} \rightarrow \frac{G}{r_0\sqrt{1-v^2/c^2}}\frac{M_0}{\sqrt{1-v^2/c^2}}\frac{m_0}{\sqrt{1-v^2/c^2}} \tag{1.5}$$

or:

$$\frac{GMm}{r} \rightarrow \frac{GMm}{r\sqrt{1-v^2/c^2}^{3}} \rightarrow M\frac{\mu_{loc}}{r}\left(1+\frac{3}{2}\frac{v^2}{c^2}\right) \tag{1.6}$$

This is a differential expression relating, the velocity, and the distance to the local gravitating mass. This can now be solved for the orbital motion, without need to make assumptions about the force mass relation.

In the following it will be shown that the equations of motion produces orbital relations, equivalent to the weak field GR relations, with the same perihelion advance:

Noting. Eq.(1.6), and inserting into Eq. (1.2), yields:

$$M_0\left[1-\left(\frac{\mu_{loc}}{r}+\frac{\mu_{loc}}{r}\frac{3}{2}\left(\frac{v}{c}\right)^2\right)-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}\right]\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]=M \tag{1.7}$$

Noting that there is only one significant cross term this becomes:

$$M_0\left[1-\frac{\mu}{r}-\frac{3}{2}\frac{\mu}{r}\frac{v^2}{c^2}-\frac{\mu^2}{2r^2}-\left(\frac{1}{2}\frac{\mu}{r}\frac{v^2}{c^2}\right)+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}\right]=M, \tag{1.8}$$

which separates into:

$$-\frac{M-M_0}{M_0}c^2=c^2\left(-\frac{\mu}{r}-\frac{1}{2}\frac{\mu^2}{r^2}+\frac{1}{2}\frac{v^2}{c^2}\left(1-4\frac{\mu}{r}+\right)+\frac{3}{8}\frac{v^4}{c^4}\right) \tag{1.9}$$

Setting the left term in this to $\in$, and note that in a conservative system, this term is constant. That is because M_0 is a defined constant and the total energy is constant.

Using the procedures for finding as outlined in Robertson & Noonan,[4] the perihelion precession, in agreement with GR is:

$$\sigma=\left(\frac{1}{2}\mu u_0-\frac{3}{2}\mu u_0+2\mu u_0+2\mu u^2\right)=\left(\mu u_0+2\mu u^2\right)\sim 3\frac{\mu}{p} \tag{1.10}$$

The detailed calculations for this are included in Appendix I.

q.e.d. It has been shown that the proper orbital equations can be derived without resorting to Riemannian spacetime.

Photon Energy

From the defining relation of this theory Eq.(1.2), the view of the Pound-Rebka-Snider[6], Mossbauer effect experiment (1960–1965)[6] changes. Instead of the photon losing energy as the photon rises in the tower, the emission of the photon at the bottom of the tower is from a less massive generator, and at a lower frequency. The generated frequency plus the added Doppler frequency provided by the velocity of the source in the experiment equals the frequency at the top, thus the photon loses no energy in the flight up the tower.

$$\nu_B\left(1-\frac{\mu}{r}\right)+\nu_D=\nu_T\,. \tag{1.11}$$

This is a departure from General Relativity. GR requires a photon escaping from a gravitational field to lose energy to the field, and in the case of a black hole the entirety of the energy is lost before escapement. Since the energy in discussion here is localized and not lost to the gravitational field, the Schwarzschild radius is no barrier.

Proper Deflection and Velocity of Light

The well known Shapiro velocity of light is arrived at by Blandford, and others [7],[14],[15], by solving the GR field equations for a constant time scale, thus giving the apparent velocity of light in three space and the utility of Fermat's principle to project ray traces.

$$c=c_0\left(1-2\frac{\mu}{r}\right), \tag{1.12}$$

Using the concept of locally conserved energy, and knowing that a photon detected at a lower position seems to gain energy from the potential energy such that:

$$E'=\frac{E_0}{\left(1-\frac{\mu}{r}\right)}\;, \tag{1.13}$$

then the local speed of light can be deduced.

In the locally conserved system it is the measurement system that has a lower rest mass and thus the photon only appears to have gained energy.

As a photon enters a gravitational field it is presumed here that the energy is not changed. The relation expected to be true is:

$$E = h\nu = h\frac{c}{\lambda} \tag{1.14}$$

From the exterior observation point the frequency does not change. This can be easily understood by observing the phase of a radio wave transmitted from space to the surface and reflected back. The number of waves and the frequency of the returning signal is the same as the transmitted signal. Thus in Minkowski space, the interior and the exterior frequency are the same and constant.

From observations it is known that in the local frames as a photon descends there is an apparent frequency increase due to an apparent increase in energy, thus:

$$E = \frac{h\nu'}{\left(1 - \frac{\mu}{r}\right)} \tag{1.15}$$

As the particle descends, there is a change in the locally observed value of the frequency, where the prime indicates a local value. GR would ascribe this to the effect of time dilation.

In the local frame;

$$\nu' = \frac{c'}{\lambda'} \tag{1.16}$$

In addition, the wavelength of a photon arriving at a local location in a gravitational field is decreased by:

$$\lambda' = \lambda\left(1 - \frac{\mu}{r}\right) \tag{1.17}$$

Putting Eq.(1.17), into Eq.(1.16), then that into Eq.(1.15), Eq.(1.14), gives:

$$E = h\frac{c'}{\lambda\left(1 - \frac{\mu}{r}\right)\left(1 - \frac{\mu}{r}\right)} \tag{1.18}$$

Comparing Eq.(1.18), to Eq.(1.14), shows are equivalence if the local speed of light at the local location is:

$$c' = c\left(1-\frac{\mu}{r}\right)^2 \qquad (1.19)$$

Comparing Eq.(1.19), with the GR equivalent, Eq.(1.12), shows them to be nearly the same accept that Eq.(1.19), has a second order term. The difference will only be detectable for motion of photons very near the gravitational radius, and thus observation of the photons in that proximity will distinguish the proper theory.

In simplistic terms when the photon enters a gravitational field, a local receiver, along the path finds the frequency goes up and the wavelength goes down, to accommodate both these effects from an external perspective in which the frequency stays constant, the speed of light must decrease according to Eq.(1.19), or the index of refraction is:

$$\eta = \eta_0 / \left(1-\mu/r\right)^2 \qquad (1.20)$$

Whereas the GR index would be established by the Shapiro Velocity with index of refraction:

$$\eta = \eta_0 / \left(1-2\mu/r\right) \qquad (1.21)$$

The second order differences in these two expressions should be soon testable by the Event Horizon Telescope [16].

CONCLUSION

With simple assumptions regarding the relation between rest mass, and relativistic mass, proper gravitational dynamics and stellar deflection phenomena can be predicted. The proposed theory yields the proper orbital equations, with the proper perihelion advance, deflection of light and gravitational red shift. The gravitational potential exchanges no energy with photons, thus photons are not bound in a gravitational field, and there are no black holes, a belief often expressed by Einstein [13]. Precision light deflection experiments near large masses, or discoveries of neutron star masses larger than GR allows, will validate or invalidate this theory.

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Appendix I

Details of Perihelion Advance

The general rest mass velocity relation proposed is:

$$M^2\left(1-\frac{v^2}{c^2}\right)=M_0^2\left(1-2\frac{\mu}{r}\right) \tag{2.1}$$

Where the velocity invariant potential is:

$$M_0^2\left(1-2\frac{\mu}{r}\frac{1}{\sqrt{1-v_2^2/c_2^2}^3}\right)=M^2\left(1-\frac{v^2}{c^2}\right) \tag{2.2}$$

Taking square root:

$$M_0\left(1-2\frac{\mu}{r}\frac{1}{\sqrt{1-v_2^2/c_2^2}^3}\right)^{1/2}\left(1-\frac{v^2}{c^2}\right)^{-1/2}=M \tag{2.3}$$

Binomial expansions:

$$\begin{aligned}(1-x)^{1/2} &= 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}\\ (1+x)^{-1/2} &= 1-\frac{x}{2}+\frac{3}{8}x^2\\ (1-x)^{-1/2} &= 1+\frac{x}{2}+\frac{3}{8}x^2+\frac{5}{16}x^3\end{aligned} \tag{2.4}$$

The simple expansion would be:

$$\left[1-2\frac{\mu_{loc}}{r_{loc}}\right]^{1/2}=\left[1-\frac{\mu_{loc}}{r_{loc}}-\frac{1}{2}\frac{\mu_{loc}}{r_{loc}}\frac{\mu_{loc}}{r_{loc}}\right] \tag{2.5}$$

Expanding all the terms in Eq. (2.3).

$$M_0\left(\left(1-\frac{\mu_{loc}}{r_{loc}}\left(1-v_2^2/c_2^2\right)^{-3/2}-\frac{1}{2}\frac{\mu}{r}\frac{\mu_{loc}}{r_{loc}}\right)\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{2}\left(\frac{v}{c}\right)^4\right)\right)=M\,, \qquad (2.6)$$

and:

$$M_0\left(\left[1-\left(\frac{\mu_{loc}}{r_{loc}}+\frac{\mu_{loc}}{r_{loc}}\frac{3}{2}\left(\frac{v}{c}\right)^2\right)-\frac{1}{2}\frac{\mu}{r}\frac{\mu_{loc}}{r_{loc}}\right]\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]\right)=M \qquad (2.7)$$

There is only one cross term of significant value.

v

$$M_0\left[1-\frac{\mu}{r}-\frac{\mu}{r}\frac{3}{2}\left(\frac{v}{c}\right)^2-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}-\left[\frac{\mu}{r}\frac{1}{2}\left(\frac{v}{c}\right)^2\right]+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]=M \qquad (2.8)$$

Simplifying and separating the mass terms:

$$M_0\left[1-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4-\frac{4}{2}\frac{\mu}{r}\left(\frac{v}{c}\right)^2\right]=M$$

$$M_0\left[1-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=M$$

$$M_0+M_0\left[-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=M \qquad (2.9)$$

$$\frac{M-M_0}{M_0}-\left[-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=0$$

multiplying by c^2, & noting that in a conservative system where the total energy is constant, the mass term is constant.

$$c^2\frac{M-M_0}{M_0}=\in \qquad (2.10)$$

Thus:

$$2\in-\left[-2c^2\left(\frac{\mu}{r}+\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}\right)+v^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right] \qquad (2.11)$$

The corresponding GR term per Robertson & Noonan.

$$2\in+\frac{2\mu c^2}{r}-v^2+\frac{2h^2\mu}{r^3} \qquad (2.12)$$

Some conventional coordinate transformations:

$$\begin{bmatrix} u-1/r & u^2\left(r^2\dot{\theta}\right)^2 = u^2h^2 & \left(\frac{dr}{dt}\right)^2 = h^2\left(\frac{du}{d\theta}\right)^2 \\ v^2 = \left[\left(\frac{dr}{dt}\right)^2 + u^2\left(r^2\dot{\theta}\right)^2\right] = \left[h^2\left(\frac{du}{d\theta}\right)^2 + u^2h^2\right] \end{bmatrix} \quad (2.13)$$

making the substitutions, we have:

$$2\in -\left(-2c^2\left(1+\frac{1}{2}\mu u\right)\mu u + v^2\left(1-4\mu u\right) + v^2\frac{3}{4}\left(\frac{v}{c}\right)^2\right) \quad (2.14)$$

Now taking the derivative with respect to the angular coordinate:

$$\frac{d}{d\theta}\left[2\in -\left(-2c^2\left(1+\frac{1}{2}\mu u\right)\mu u + \left[\left(\frac{dr}{dt}\right)^2 + u^2\left(r^2\dot{\theta}\right)^2\right]\left(1-4\mu u\right) + v^2\frac{3}{4}\left(\frac{v}{c}\right)^2\right)\right], (2.15)$$

or:

$$\frac{d}{d\theta}\left[2\in -\left(\begin{array}{l} -2c^2\left(1+\frac{1}{2}\mu u\right)\mu u \\ +\left[\left(\frac{dr}{dt}\right)^2 + u^2h^2\right]\left(1-4\mu u\right) \\ +\ v^2\frac{3}{4}\left(\frac{v}{c}\right)^2 \end{array}\right)\right] \quad (2.16)$$

Making some substitutions.

$$0 = \frac{d}{d\theta}\left[2\in -\left(\begin{array}{l} -2c^2\left(1+\frac{1}{2}\mu u\right)\mu u \\ +h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]\left(1-4\mu u\right) \\ +\frac{3}{4}\frac{h^4}{c^2}\left[h^2\left(\frac{du}{d\theta}\right)^2 + u^2\right]^2 \end{array}\right)\right] \qquad \frac{\left(\frac{dr}{dt}\right)^2 = h^2\left(\frac{du}{d\theta}\right)^2}{v^2 = \left[h^2\left(\frac{du}{d\theta}\right)^2 + u^2h^2\right]} \quad (2.17)$$

Differentiating the three terms, designating each as A,B, & C:

$$\left[\frac{d}{d\theta}\left(-2c^2\left(1+\frac{1}{2}\mu u\right)\mu\right) = -2\mu c^2\left(1+\mu u\right)\frac{du}{d\theta} = -2h^2\frac{du}{d\theta}\frac{\mu c^2}{h^2}\left(1+\mu u\right)\right] \quad (A)$$

Parts of the B term:

$$\left[\begin{array}{l} \frac{d}{d\theta}h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] \\ = h^2\left[2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2}\right) + 2u\left(\frac{du}{d\theta}\right)\right] \\ = \left[2h^2\left(\frac{du}{d\theta}\right)\right]\left[\left(\frac{d^2u}{d\theta^2}\right) + u\right] \\ \\ \frac{d}{d\theta}\left(1-4\mu u\right) = -4\mu\left(\frac{du}{d\theta}\right) \end{array}\right] \qquad (2.18)$$

So the B term is:

$$\left[\begin{array}{l} \frac{d}{d\theta}\left\{h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]\left(1-4\mu u\right)\right\} \\ = \left[\begin{array}{l} -h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]4\mu\left(\frac{du}{d\theta}\right) \\ +\left[2h^2\left(\frac{du}{d\theta}\right)\right]\left[\left(\frac{d^2u}{d\theta^2}\right) + u\right]\left(1-4\mu u\right) \end{array}\right] \\ = 2h^2\left(\frac{du}{d\theta}\right)\left[\begin{array}{l} \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]\left(-2\mu\right) \\ +\left[\left(\frac{d^2u}{d\theta^2}\right) + u\right]\left(1-4\mu u\right) \end{array}\right] \end{array}\right] \quad (B) \qquad (2.19)$$

And the C term:

$$\left[\begin{aligned} &\frac{d}{d\theta}\ \frac{3}{4}\frac{h^4}{c^2}\left[h^2\left(\frac{du}{d\theta}\right)^2 + u^2\right]^2 \\ &\qquad = \frac{3}{4}\frac{1}{c^2}h^2h^2\,2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)\frac{d}{d\theta}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) \\ &\qquad = \frac{3}{4}\frac{1}{c^2}h^2h^2\,2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)\left(2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2}\right) + 2u\left(\frac{du}{d\theta}\right)\right) \\ &\qquad = 2h^2\left(\frac{du}{d\theta}\right)\frac{3}{2}\frac{1}{c^2}h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)\left(\left(\frac{d^2u}{d\theta^2}\right) + u\right) \\ &\qquad = \left(2h^2\left(\frac{du}{d\theta}\right)\right)\frac{3}{4}\left[2\frac{1}{c^2}h^2u^2\right]\left(\left(\frac{d^2u}{d\theta^2}\right) + u\right) \qquad \left(\frac{du}{d\theta}\right)^2 \sim 0 \end{aligned}\right] \quad (C)(2.20)$$

Collecting and factoring a common term gives:

$$0 = -2h^2\left(\frac{du}{d\theta}\right)\left[\begin{array}{ll} - & \frac{\mu c^2}{h^2}\ (1+\mu u) \\ + & \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right](-2\mu) \\ + & \left[\left(\frac{d^2u}{d\theta^2}\right) + u\right](1-4\mu u) \\ + & \left[\left(\frac{d^2u}{d\theta^2}\right) + u\right]\frac{3}{2}\left[\frac{h^2u^2}{c^2}\right] \end{array}\right] \qquad (2.21)$$

collecting common terms reduces the number of terms:

$$0 = \begin{bmatrix} - & \dfrac{\mu c^2}{h^2}\left(1+\mu u\right) \\ + & \left[\left(\dfrac{du}{d\theta}\right)^2 + u^2\right]\left(-2\mu\right) \\ + & \left[\left(\dfrac{d^2u}{d\theta^2}\right)+u\right]\left(1+\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]-4\mu u\right) \end{bmatrix} \tag{2.22}$$

Dividing by the coefficient of the second order term gives:

$$0 = \begin{bmatrix} -\dfrac{\mu c^2}{h^2}\left(1+\mu u\right)/\left(1+\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]-4\mu u\right) \\ -\left[\left(\dfrac{du}{d\theta}\right)^2 + u^2\right]\left(2\mu\right)/\left(1+\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]-4\mu u\right) \\ +\left[\left(\dfrac{d^2u}{d\theta^2}\right)+u\right] \end{bmatrix} \tag{2.23}$$

or:

$$0 = \begin{bmatrix} -\dfrac{\mu c^2}{h^2}\left(1+\mu u-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+4\mu u\right) \\ -\left(2\mu u^2\right)\left(1-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+4\mu u\right) \\ +\left[\left(\dfrac{d^2u}{d\theta^2}\right)+u\right] \end{bmatrix} \qquad \left(\dfrac{du}{d\theta}\right)^2 \sim 0 \tag{2.24}$$

The equation for a circle is:

$$\left(\frac{d^2u}{d\theta^2}\right)+u = u_0 + f \tag{2.25}$$

Where f is a perturbation of the orbit.

The precession, per the procedure of Robertson & Noonan is $\sigma = \frac{1}{2}\frac{\partial}{\partial u} f$.

Where in this case f is:

$$f = \begin{bmatrix} u_0 \left(+\mu u - \frac{3}{2}\left[\frac{h^2 u^2}{c^2}\right] + 4\mu u \right) \\ \left(2\mu u^2 - 3\mu \left[\frac{h^2 u^4}{c^2}\right] + 8\mu^2 u^3 \right) \end{bmatrix} \tag{2.26}$$

Where: $\frac{\mu c^2}{h^2} = u_0 = \frac{1}{p\left(1-e^2\right)}$.

So:

$$\sigma = \frac{1}{2}\frac{\partial}{\partial u}\begin{bmatrix} u_0 \left(+5\mu u - \frac{3}{2}\left[\frac{h^2 u^2}{c^2}\right] + 2\mu u^2 \right) \\ +\left(-3\mu \left[\frac{h^2 u^4}{c^2}\right] \quad +8\mu^2 u^3 \right) \end{bmatrix} \tag{2.27}$$

Where σ is a ratio of the perihelion advance to the orbit circumference

$$\sigma = \frac{1}{2}\left[\left(\begin{matrix} \left(+5\mu u_0 - 3\left[\frac{h^2}{c^2}\right] u_0 u + 4\mu u \right) \\ +\left(-12\mu \left[\frac{h^2 u^3}{c^2}\right] + 24\mu^2 u^2 \right) \end{matrix}\right)\right] \qquad \frac{\left[\frac{h^2}{c^2}\right] = \frac{\mu}{u_0}}{\left[-\frac{1}{2}12\mu\left[\frac{\mu}{u_0}\right]u^3 + \frac{1}{2}24\mu^2 u\right] \sim 0} \tag{2.28}$$

Then we have for the precession:

$$\sigma = \left(\frac{1}{2}\mu u_0 - \frac{3}{2}\mu u_0 + 2\mu u_0 + 2\mu u^2 \right) = \left(\mu u_0 + 2\mu u^2 \right) \sim 3\frac{\mu}{p} \tag{2.29}$$

The units are the ratio of the advance to the orbital circumference.

Comparing with the GR value from Robertson & Noonan:

$$\sigma = \frac{1}{2}\frac{\partial}{\partial u}\left(\left(\frac{dr}{dt}\right)^2 2\frac{u}{c^2} + 2\frac{uu^2}{c^2} h^2\right) = 3\frac{u^2}{c^2} h^2 = 3\mu u\left(\frac{u}{u_0}\right) \qquad (2.30)$$

Thus our procedure yields the proper perihelion precession.